

“THE DIRAC FIELD”

Where the fun begins…



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LORENTZ TRANSFORMATIONS



Under Lorentz transformation:



now, this is for a real scalar field.

In general, a field under LT transforms as:



Where, form a representation of Lorentz group.

* and

These are some of the conditions that are representation of Lorentz group must follow.

“**BIT ON LORENTZ GROUP**”

Lorentz group is a non-compact group. It is a S0(3,1) group meaning it is a set of 4\*4 orthagonalmatrices with real matrix elements and determinant being +1 satisfying :-



,  
where, is a 4\*4 diagonal matrix a.k.a our goodol’ Minkowski metric .

This group is generated by 6 matrices. We first list 3 rotation generators of Lorentz group:

Where rotation generators satisfy:

Where is our old friend from when we generalizes cross product, levi-civita symbol.

Sly. We can find the boost generators of Lorentz group:

Sly. They also satisfy a commutation relation:

These are 6 generators of our SO(1,3) group.

S0(1,3) is also a lie group, you can check it for yourself. I’m not gonna do it here.

Now, we look at infinitesimal transformation of Lorentz group and study the resulting the lie algebra…



Where “w” is an infinitesimal.

i.e. it is a anti-symmetric matrix.

This 4\*4 anti-symmetric metric has 6 independent components. 3 for each rotation and boots.

Now, we’ll introduce the basis of these 6, 4\*4 matrices. ; A = 1,…,6.

Now, we’ll do a bit of trickery and replace A with 2 anti-symmetric indices.

Quick FYI: for practical uses we’ll lower one index of M, let’s say the νthindex.lowering index leads to the entry of more minus signs, which messes with our anti-symmetry.



Ex: generates boost along direction. Real and symmetric



Lie Algebra:

Again, we use a bit of trick in order to write our “w” matrix comfortably.



| where, ’s only help in telling us which transformation that we are performing.Now, we use exponential maps for the finite Lorentz transformation, to represent Λ:



are the generators of the Lorentz transformation.

**SPINOR REPRESENTATION :**

We saw that we weren’t able to describe fermions, eg:- electrons with our lagrangian for scalar field. But ,we know that this much is true that, “they should obey special relativity” and for that we need Lorentz invariance. So, that’s why need another representation of Lorentz group.



CLIFFORD ALGEBRA:

; μ,ν = 0,1,2,3,4

Where ’s are our old friend from QM, Pauli matrices.

And they also anti-commute:



You must remember this pauli matrices stuff from your QM course or whatever.

You can construct many other representations of Clifford algebra by taking , for any matrix V(If it’s IN physics it’s INvertible)…jk do this for only invertible matrices V.

By, now you guys must’ve gotten some sort intuition that these gamma matrices are the basis for our new representation of Lorentz group. Well, you’re not all wrong

We define:

this is a commutator that we might need a bit later.

These matrices are anti-hermitian i.e. =

Now, for the most important question does this qty. “S” that we defined earlier, even form a representation of Lorentz group?-Indeed it does, why else would I spend time introducing it then. You can check the following commutator:



If I’ve fucked up the indices here, GOD strike me dead(weird flex, IK).

SPINORS:  
we let denote the rows and columns of by

We need to a field for to act upon. We’ll call it *dirac spinor field,* with 4 complex components a =1,2,3,4.



Now, let’s see how this dirac spinor field transforms under Lorentz transformation.

And the new representation will transform like:

Now, we’ll do some examples to show how this new representation is different from the good ol’ vector representation.



Ex: Rotations:



We can write our “rotation parameter” . Let’s perform a 2π radian rotation along x^3-axis. The spinor rotation matrix now becomes:



| under 2π rotation.



This isn’t the result that we get from “normal” vector representation.



BOOSTS:

We can do the same computation for the boost. The generators will be of the form . Taking the boost parameter to be



You guys must’ve noticed by now, that for spinor rotation matrix i.e. unitarity maintained but NOT for boosts.

CONSTRUCTING AN ACTION:

At the end of the day, what we want is Lorentz invariant EOM. To obtain that we must write an Lorentz invariant action.



Is a Lorentz scalar?

Let’s check:

But, . So, this just won’t cut it ;(



Now, let’s take a step back and see “daggers” of gamma matrices.

,

Now,

Now, we define what’s called a dirac adjoint:



Now, is a Lorentz scalar?



….Finally, some progress. We’ve constructed a Lorentz scalar.

In similar fashion, we can construct Lorentz vector which looks something like this (proving this Lorentz invariant involves some subtleties, which we won’t go over here) and transforms like a Lorentz tensor.

THE ACTION:

We’ve now constructed many Lorentz invariant by now. We can construct an action, it looks like this:



TO THE DIRAC EQUATION ASAP:

Varying action w.r.t :-

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The beauty of this formula is that each component solves the Klein-Gordon equation:



This last equation has no gamma matrices this means, it can be applied to each component of

DEGREES OF FREEDDOM:

In classical meachanics we count degrees of freedom by counting the dimension of the configuration space or half the dimension of phase space. In field theories we have infinite degree of freedom, but we can still count the DOF per spatial point. Eg: a complex scalar has 2 DOF per spatial point, which results in 2 “kinds” of particles (paricle and anti-particle).

Our Dirac spinor has 4 complex components which means 8 real components. So, are there 8 kind of particles? Well, NO!Let’s look at the conjugate momenta to dirac spinor :



This means phase space for dirac spinors is parametrized by unlike real and complex scalar fields. So, the phase space for dirac spinor has 8 real components this means it has 4 DOF. Good, we’ve made some progress. So, this means are there 4 kinds of particles? HELLS NAH!. We still has some work to do, but unfortunately it’s out of the range of this talk. It’s covered, when we start to quantize the dirac spinor field, and we finally reduce it to 2 DOF which corresponds to a particle and an anti-particle.

